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PG I semester

Paper - CC-2

Unit - 2

Topic - Evaluation of difinite integrals

* Evaluation of Definite integrals

By the use of Cauchy Residue theorem certain type of definite integrals may be evaluated. In each given case the choice of a suitable curve along which the integration is to be affected; plays an important role. This curve is usually known as a contour and the integration along the same is called Contour Integration. It may be observed that a definite integral that can be evaluated by the use of Cauchy residue theorem may be evaluated by other methods although not so easily. While evaluating the integrals by Cauchy residue theorem,

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it is sometimes convenient to use the following two important theorems.

<u>Two Important theorems</u>:- In the evaluation of integrals by the method of complex variables, it is sometimes convenient to use the following theorems

<u>Theorem ①</u> If AB is an arc of a circle $|z| = R$ having $\theta_1 \leq \theta \leq \theta_2$ and $\lim_{R\to\infty} z f(z)$ tends uniformly to b; then

$$\lim_{R\to\infty} \int_{AB} f(z)\,dz = ib(\theta_2 - \theta_1) \quad \text{--- ①}$$

Proof:- Consider an arc of a circle $|z| = R$ having $\theta_1 \leq \theta \leq \theta_2$. Given $\lim_{R\to\infty} z f(z) \to b$.

Choosing the radius the circle $|z| = R$ sufficiently great, we can make

$|z f(z) - b| < \epsilon$, where ϵ is very small number.

This implies

$$z f(z) = b + \eta \quad \text{where } |\eta| < \epsilon$$

i.e.,
$$f(z) = \frac{b+\eta}{z}$$

$$\therefore \int_{AB} f(z)\,dz = \int_{AB} \frac{b+\eta}{z}\,dz$$

Substituting $z = Re^{i\theta}$: so that $dz = Re^{i\theta} i\,d\theta$

we get

$$\int_{AB} f(z)\,dz = \int_{\theta_1}^{\theta_2} (b+\eta) i\,d\theta = ib\int_{\theta_1}^{\theta_2} d\theta + \int_{\theta_1}^{\theta_2} \eta\, i\,d\theta$$

$$\therefore \int_{AB} f(z)\,dz = ib(\theta_2 - \theta_1) + \int_{\theta_1}^{\theta_2} \eta\, i\,d\theta$$

$$\therefore \left| \int_{AB} f(z)\,dz - ib(\theta_2 - \theta_1) \right| \le \int_{\theta_1}^{\theta_2} |\eta|\,|i\,d\theta| < \epsilon(\theta_2 - \theta_1)$$

In the limit $R \to \infty$, $\epsilon \to 0$,

therefore we have

$$\lim_{R\to\infty} \int_{AB} f(z)dz - ib(\theta_2 + \theta_1) = 0$$

$$\therefore \lim_{R\to\infty} \int_{AB} f(z)dz = ib(\theta_2 - \theta_1)$$

Proved

Theorem 2:- If AB is the arc of a circle $|z - z_0| = r$, having $\theta_1 \le \theta \le \theta_2$ and $\lim_{z\to z_0} (z - z_0) f(z) = b$; where b is a constant; then

$$\lim_{r\to 0} \int_{AB} f(z)dz = ib(\theta_2 - \theta_1) \quad \text{--- (2)}$$

Proof:- consider an arc of a circle $|z - z_0| = r$ having $\theta_1 \leq \theta \leq \theta_2$.

Given $\lim\limits_{z \to z_0} (z - z_0) f(z) = b$

Choosing the radius of the circle $|z - z_0| = r$ sufficiently small, we can make

$$|(z - z_0) f(z) - b| < \epsilon \text{ for } |z - z_0| < \delta$$

where δ is very small. choosing ϵ less than δ, we can write

$$(z - z_0) f(z) = b \quad \text{where } |\eta| < \epsilon$$

i.e., $$f(z) = \frac{b + \eta}{z - z_0}$$

$$\therefore \int_{AB} f(z)\, dz = \int_{AB} \frac{b + \eta}{z - z_0}\, dz$$

Putting $(z-z_0) = re^{i\theta}$ i.e.,

$dz = re^{i\theta} i\, d\theta$, we have

$$\int_{AB} f(z)\,dz = \int_{\theta_1}^{\theta_2} \frac{(b+\eta)\, re^{i\theta}}{re^{i\theta}}\, i\, d\theta$$

$$= bi\int_{\theta_1}^{\theta_2} d\theta + \int_{\theta_1}^{\theta_2} \eta\, i\, d\theta$$

$$= bi(\theta_2-\theta_1) + \int_{\theta_1}^{\theta_2} \eta\, i\, d\theta$$

This gives

$$\left|\int_{AB} f(z)\,dz - ib(\theta_2-\theta_1)\right| \leq \int_{\theta_1}^{\theta_2} |\eta|\,|i|\,|d\theta| < \epsilon(\theta_2-\theta_1)$$

In the limit $r \to 0$, $\epsilon \to 0$; therefore

$$\lim_{r\to 0}\int_{AB} f(z)\,dz = ib(\theta_2-\theta_1)$$

Proved